WHAT DO WE KNOW ABOUT MATHEMATICS CURRICULA?

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WHAT DO WE KNOW ABOUT MATHEMATICS CURRICULA?

The subject of this essay is the mathematics curriculum: What should we be teaching in mathematics, and in what ways? This issue has been a focus of my problem solving work for nearly two decades, and I have written about it at length from that perspective. However, I am going to take a different point of view in this essay. Here I shall take a distanced perspective, in order to reflect on some difficult issues. Mathematics education is at a turning point. Some radically new programs are being proposed, and the abolition of some familiar programs is being proposed as well. This is a good time to ask, What do we really know? How much of what we think we know is based on a firm knowledge base, how much on informed guesswork, how much is really just opinion? How much of what we plan to do reflects cultural biases, rather than established fact? These are thorny questions. I shall explore the following four major issues related to curriculum: questions of content, tracking, problem-based curricula, and the role of proof. My goal is to be as honest about what I know, and what I don't know, as I can be.

THE QUESTION OF CONTENT

What do we want students to learn in mathematics courses, and how do we characterize what they have learned? I begin with a story that casts the issue in sharp relief. The story is true, with proper names replaced by variables because the specifics of people and place are irrelevant – and because the conversation described might well have taken place at dozens of universities around the country.

When I joined the mathematics department at University X some years ago I had a chat with the department chair, Professor Y, about my problem solving course. Y was truly supportive. He endorsed the idea of offering the course and helped me decide the level at which it should be offered. (Previous incarnations of the course had been given for student groups that ranged from first year liberal arts majors to advanced mathematics majors.) We decided the course should be offered as a lower division elective, so that it would be accessible both to potential mathematics majors and to non-majors who might want a dose of mathematical thinking above and beyond formal course requirements. After we had assigned it a course number, and right after Professor Y mentioned that the course might become an attractive option for mathematics majors, the following dialogue took place.

Y: Of course we can't give credit toward the major for taking the course.

AHS: Why not?

Y: Because you're not teaching content.

I said that there was indeed content to the course: In solving the problems I assigned in the course discussions the students did a reasonable amount of elementary number theory, geometry, and so on. But Y didn't buy the argument, because the content wasn't identified and packaged in the standard ways; I didn't cover specific theorems, bodies of knowledge, etc. (And, it is true that the students and I work through material in the problem solving course at a much slower pace than we would in a lecture class.) So I tried a different tack.

AHS: I'll tell you what. The goal of the course is simple: I want the students to be able to solve problems I haven't explicitly taught them how to solve. The real test is my final exam, which is a collection of difficult problems that don't look very much like the ones I discuss in the course. Here's a bet. Suppose I have 20 kids in the class, just the ones who've enrolled for it. You can hand pick 20 junior and senior math majors and give them the exam. I'll bet you my semester's salary that my kids will outperform yours on the final.

Y: I'm sure you're right, but we still can't give credit toward the major. You're not teaching them content – you're just teaching them to think.

What can one say? Only that three years later Professor Z, chair of the computer science department, asked me if I would be willing to have his department require my course of all their majors. It was the only elementary mathematics course in which they learned anything, he said, and the department was interested in the possibility of removing the calculus requirement and requiring my course instead. (Since I was about to leave for another university, the swap didn't happen.)

The moral of this story and the reason that I tell it is that it demonstrates clearly that what counts as mathematical "content" depends on one's point of view. From Professor Y's perspective, the mathematical content of a course (or a lesson, or a whole curriculum) is the sum total of the topics covered. This, of course, is the traditional view. It lies behind the use of standard "scope and sequence" charts to characterize school curricula, the traditional "Math 9 is Algebra I, Math 10 is Geometry, and Math 11 is Algebra II/Trig" curriculum labels, and, for example, the delineation of a proposed new course outline for Math 1A (first-semester majors' calculus) at Berkeley: "Limits and rate of change, 4 lectures; derivatives, 7 lectures; exp, log, and inverse trigonometric functions, 8 lectures. . . ." It is functional in curricular terms, because it fits with the standard, hierarchical notion of curriculum structure: Topics in course A are prerequisites for course B, and so on. This point of view is familiar. I grew up with it, as did the vast majority of my colleagues. It is comfortable, and it is dangerous.

The danger in this kind of "content inventory" point of view comes from what it leaves out: the critically important point that *mathematical thinking* consists of a lot more than knowing facts, theorems, techniques, etc. This understanding was at the heart of Pólya's work, and it has been significantly elaborated over the past two decades; see Schoenfeld (1992a) for a review. In a nutshell, the emerging view of mathematics learning differs significantly in perspective, scope, and detail from the traditional view.

In terms of overall perspective, the difference can be captured in a simple phrase: I would characterize the mathematics a person understands by describing what that person can *do* mathematically, rather than by an inventory of what the person "knows." That is, when confronted with both familiar and novel situations that call for producing or using mathematical ideas, can someone do so effectively? Note that this performance standard is the one that Professor Y lives by in his professional life, and the one that he uses to judge his colleagues. University mathematicians get promotion and tenure by producing new mathematics – i.e., by developing new or deeper understandings, seeing new connections, solving unsolved problems, etc. In or out of the university, mathematicians earn their keep by posing problems and solving them, either by developing new approaches or by recognizing that a new problem, looked at in the right way, is closely related to a familiar (and solvable) one. The key point here is that mathematicians have to *use* what they know; just knowing isn't enough. It is ironic that this is precisely the standard I proposed for the evaluation of my problem solving class – could my students outperform others, who had far more training? – and that he rejected, because it didn't fit with his notion of content. He was happy to agree that I taught students to think mathematically, and impressed by the students' work: My students have produced results that led to a paper published in the *College Mathematics Journal* (Schoenfeld, 1989), also some minor results in number theory that are unfamiliar to most mathematicians (Schoenfeld, 1990). But, trapped by a view that – at least in classroom settings – understanding should be measured in terms of "the quantity of recognizable mathematics known," he had no way to assign value to the things my students had learned to *do*.

Issues of scope and detail are far too complex to deal with here, except in the most cursory fashion; see Schoenfeld (1985; 1992a; in press) for detail. Suffice it to say that when one focuses on mathematical thinking – the ability to do or use mathematics – that one needs to pay attention to (i) content, classically conceived[1]; (ii) problem solving strategies or heuristics; (iii) control, which is concerned with how well and how efficiently people use the mathematical resources at their disposal; (iv) beliefs; and (v) one's ability to function as a member of a mathematical community (e.g. the ability to communicate with others with and about mathematics). When your goal is that students learn to think mathematically (rather than merely mastering content), you have to attend to all of these, and you have to foster and reward them.

In my problem solving courses I try to encourage and reward *doing* mathematics as described above, and content in the standard sense gets short shrift. An argument can be made that, in a variety of ways, the courses are quite successful. Would I recommend that all mathematics curricula should look like age-appropriate

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[1] "Classically conceived" may be the wrong phrase here. As Dick Stanley points out (personal communication, March 24, 1993), one can conceptualize a body of mathematical content as more or less rote, mechanical procedures, or one can see it as a rich, deeply connected collection of ideas. The latter is preferable, though rare. As a community, we may well need to expand our notion of what it means to understand content per se. For a discussion of the issue in the case of linear functions, see Moschkovich, Schoenfeld, and Arcavi (1993).

versions of the problem solving courses? If so, what justification would I have for that recommendation? And if not, just what is justified?

I would not – at least not without significant qualifications. Before I deal with those, it is important to understand the context within which I give my problem solving courses. I teach at the college level. My students are the successes of our system, which means they have, for the most part, had twelve years or more of instruction that focuses largely on mathematical content and have had almost no exposure to the process aspects of mathematical thinking. Their experiences are analogous to those of students who have had significant training with wood shop tools (lathe, band saw, joiner, sander, etc.) but have only practiced on scrap wood and never had the opportunity to build something using their skills. In the problem solving course, I am making up for twelve years of process deficits in their education; while there are more than occasional content deficits, I can be assured that content has received major attention in their prior mathematics courses, and that it will be the major focus of any other courses that they take in our mathematics department. Given that, I can afford to relax content concerns somewhat.

However, the issues are different if one is to design a whole curriculum from scratch. Then, you're accountable for all of the content. What role would content and process play in those circumstances? And what would the style of instruction be?

There are some ways in which I feel comfortable answering these questions, others in which I do not. Things about which I am confident: (1) most mathematics can be taught in the style of my problem solving courses; (2) large amounts of mathematics can be learned as sensible answers to sensible questions – i.e., as part of mathematical sense-making, rather than by "mastery" of bits and pieces of knowledge; (3) many basic skills can be picked up in the context of meaningful mathematical work. Things about which I am not confident: (1) how much mastery of some basics is required for competent, flexible performance on more demanding tasks; (2) what the best ways of mastering some of those basics might be; (3) how best to think about organizing a curriculum in a way that does justice to what is important in the traditional content, while engaging students meaningfully with the mathematics. I do have one confident "bottom line," discussed below.

Let me begin with the items on the confident side of the ledger. I find it interesting to note that as my problem solving course developed over the years, I wound up teaching all of my other mathematics courses quite differently. It's not that I set out to teach problem-solving heuristics, control, beliefs, and so on; when I taught standard courses, my goals were to teach course content (and in multiple-section courses, my students needed to be prepared to take common final exams). What I did do, however, was spend more in-class time thinking carefully about fewer things. Instead of saying "Here's a class of problems and here's the smart way of dealing with them" – the standard lecture technique – I would say "Here's a problem people find important. How can we make sense of it?" The class and I would then try to do so.

This approach works at many levels. At the level of a single class, one often finds that formulas which strike students as coming out of the blue can instead be presented in such a way that students find them perfectly natural. One trivial example: in a pre-calculus course I had assigned my students the section of their text that discusses the fact that in a circle of radius $\mathbf{r}$, a central angle of measure $\boldsymbol{\theta}$ subtends an arc of length $\mathbf{s} = \mathbf{r}\boldsymbol{\theta}$. The students said they had a hard time with the text; virtually none raised their hands when I asked how many had understood it. But when I asked "Suppose you had a circle of radius 7, and a right angle at the center. What is the length of the part of the circumference that is cut off by that angle?" the students had no difficulty whatsoever; after a few similar examples, they made the appropriate generalizations, and derived the appropriate formula.

The same approach works equally well on the scale of a two-week or month-long unit. When I taught the "learning to teach mathematical thinking" course in Berkeley's teacher preparation program, the course project was for students to take a unit that they were going to teach toward the end of the year, reconceptualize it, and then teach it in such a way that the content emerged as a set of plausible answers to some reasonable questions. The task was not easy for the students, but it *was* rewarding. (Note that novice teachers can be successful at this; it doesn't take an expert.) And, of course, some major course development attempts (e.g., the Grenoble group's reconceptualization of calculus as motivated by historically important problems; Alibert, 1988) are based on similar notions of coherence and sense-making in mathematics. A footnote on to basic skills: when the mathematics is meaningful and students are interested in what's going on, the students who need to brush up on their skills seem to do so without too much trouble.

The payoffs of such an approach are clear. I have written about them at length; more broadly, the pages of *JMB* though the years are chock full of examples of student work that illustrate what students can do when they are given the opportunity to truly engage in mathematics. There is, of course, a cost to having the students engage more deeply in the mathematics: one "covers" less. However, when the payoffs include much deeper understanding, much longer retention of the content, enthusiasm, and the fact that the students get a much better sense of the mathematical enterprise, the price in (ostensible) coverage is a small one to pay.

Now to the caveats. In what may be a natural reaction to years of curricula that emphasized drill-and-kill on sterile computations and symbol manipulation, many proponents of new curricula staunchly advocate taking an absolutely minimalist approach to the basics. Their justifications may be on the grounds that (a) technology will render such computational and symbolic skills irrelevant; (b) such skills will develop naturally, in context, when they are needed; or (c) students are turned off by such things, so they should be avoided. I find all three justifications problematic, and I worry about the competencies that students will need in order to develop mathematical power. Personally, I don't place much emphasis on basic algorithms, or on simple symbol manipulation capacity. However, my judgment on this issue can't be trusted: I'm fluent in those skills, so it's easy for me to discount their importance. There are at least two issues here, and we are (or at least I am) woefully ignorant about them. The first has to do with the degree of fluency required to do competent work. I certainly don't believe in hierarchies of skills, at least in the version of hierarchies that says that you have to have mastered prerequisite skills before being allowed to work on more complex tasks. But, to give an example in terms of low-level skills, I have no doubt that the reason I can factor simple polynomials without difficulty is that I can factor the constant terms in them almost reflexively. I don't have to work at factoring (say) $(x^2 - 2x - 24)$ because (a) the factors of 24 leap out at me, and (b) I can do the multiplication to check the factoring with ease. Never mind whether factoring $(x^2 - 2x - 24)$ is important; that's beside the point. *Some* such skills are important for students, if only because not to be fluent at them means that one's clumsiness at them will get in the way when one needs to see past them. I'm certain, for example, that major components of my problem solving abilities (for what they are) are my ability to see whether certain directions will turn out to be easy or hard ("this looks like it will generate three equations in three unknowns, which I can handle") and my tolerance for hack work during the "mucking around" phase of coming to grips with a problem, or while exploring avenues of solution. Now, there may well be different routes to the development of such abilities than the one I took myself, back in the drill-and-kill pre-technology days. But the question of which skills are necessary, to what degree of competency, and what kinds of problems will be caused if students are weak at them, is very much open. (Note that the question is open whether or not one assumes that students will have access to computers or calculators.)

The second issue is more subtle, and has to do with the development of intuition – with number sense for younger students, and something that might be called symbol sense for older ones. How much experience does one need to have, and of what types, to develop a feel for the mathematical objects one works with? This area is under-researched, and it is seriously under-conceptualized. We have little understanding of what something like number sense might really be (see Greeno, 1991; Schoenfeld, 1992b), much less an understanding of what kind of experiences might contribute to it in a reasonable way; and broader issues of the role of experience in the development of intuition are even more poorly understood. Anyone who claims to know the answer is, in the old phrase, a liar or a fool, or both.

At the more macroscopic level, there is a similar set of uncertainties about overall curriculum organization. Documents such as the *Standards* (NCTM, 1989, 1991) and *On the Shoulders of Giants* (Steen, 1990) reflect both the tremendous amount of progress that has been made in conceptualizing curriculum and the long way we have to go. Generations of students, teachers, and mathematics faculty in the United States grew up assuming that the identities

Math 9 = Algebra I
Math 10 = Geometry (Euclidean or otherwise, depending on the course)
Math 11 = Algebra II/Trig

were cast in stone, while they were in fact an artifact of American curricular history. Indeed, the world at large has long viewed the degree of homogeneity within each of those courses, and the absence of connections among them, as a uniquely American pathology. Integrated curricula have been the reality world-wide for a long time, and only those Americans who are ignorant or chauvinistic can reject them on the grounds that they "can't work" or that they violate sensible, evolved ways of teaching. But the question of how to mix things so that important substance doesn't get lost in the shuffle is a decidedly non-trivial issue, and one for which we don't have answers. If, for example, what we now call "algebra" is distributed through the curriculum in bits and pieces and learned in specific problem solving or applied contexts, how do we know when and to what degree students will have the relevant algebraic skills to deal with problems they will encounter? (See "Problem-Based Curricula, below.) When and where should students deal with various abstractions, with symbol manipulation, with other aspects of mathematics that, while disproportionately represented in current curricula, still represent important intellectual achievements? These are difficult questions, and my answer is the same as above: those who claim to know the answers are liars, fools, or both. The resolution of many of these issues is likely to be a long time in coming, and it is likely to be as much empirical as it is grounded in theory. The state of our theoretical understandings is much better than it was, but we still have a long way to go.

Now to the final and confident bottom line of this section. One might read the cautions and caveats expressed in the previous few paragraphs as grounds for delay or inaction, as evidence for arguments of the following type: "We don't know enough; the empirical evidence isn't in; we shouldn't try untested programs and put students at risk thereby." Indeed, "progressive" colleagues have warned me that the forces of darkness (read: "conservatives") would seize upon my caveats as excuses to sidetrack the reform movement; hence, they said, I should keep my qualms to myself. I think either extreme is just plain wrong. On the one hand, we have to recognize that the current educational system *is* an experiment – and that the evidence in favor of it is not all that favorable. If we were to eradicate local tradition as a factor and do "zero-based curriculum design" on the basis of what we now understand about thinking and learning, the odds are that the traditional American curricular design would merely be one of many competitors, and a pretty low priority one at that: while certain aspects of its organization have surface validity, what evidence there is suggests that it has a lot of drawbacks! Make no mistake, there *is* a mandate for principled change: Good ideas should be tried and given license to sink or swim. On the other hand, those who are developing new curricula should, despite reformist zeal, proceed with due caution. After all, the empirical data are not in. And, as the history of reform amply demonstrates, biases, even if well grounded, are not enough to sustain reform. We need to keep our eyes open, to improve on what works and to jettison what doesn't. This will call for unusual honesty on the part of developers – alas, the number of studies where people report honestly on the *difficulties* with their new ideas or methods is small indeed. Also, we need those who are doubtful to be critical but reasonable in their doubts and criticisms, so that the honesty of the innovators is rewarded and encouraged, rather than quickly viewed as a mistake to be avoided in the future.

TRACKING

I am a product of the tracking system. Classes were explicitly tracked (grouped by "ability") when I attended elementary and junior high school; they were implicitly tracked (one's academic interests in large measure determined one's peer group) in high school. The system worked for me, in that each academic success paved the way for the next. My school grades got me into a publicly supported college, and my record there got me accepted to graduate school with financial aid. The path was pretty smooth, and that's the way it's supposed to work. Indeed, in days gone by it appeared that the system *did* work – at least in the sense that it provided an adequate number scientists. American mathematics, science, and engineering were the envy of the world.

Times have changed. From the functional point of view, it appears that the system – which then, as now, filtered out half of the students in the mathematics and science pipeline each year from ninth grade on – is incapable of producing the quantities of trained mathematicians, scientists, and engineers we will need. Thus, on functional grounds alone, it needs to change. It certainly needs reform on the grounds of equity: There is no question that, even if inadvertently, the system has consistently and disproportionately disenfranchised minorities and women. It may have worked "democratically" and "meritocratically" for me as described in the previous paragraph, but the fact is that I was in the segment of the population (white, male, with a cultural heritage that prized education and pushed me along those lines) that stood to profit from the system. Many, many others who might have trod the same path never had the chance. It is clear that we need to expand the pool and take advantage of the huge amount of talent that lies outside the support structures of the current system. And we need to provide all students with the kinds of mathematical experiences that will induce them to go on in mathematics and science, rather than stopping as soon as possible.

Note that we need to do so not only to produce the our scientific leaders and work force, but to produce a generally scientifically literate society. When a candidate accused of "voodoo economics" can announce in the presidential debates that "the experts tell me that there's a line that goes like this [gesturing downwards to the right], and a line that goes like this [gesturing upwards to the right], and that when the lines cross, the budget will be balanced," go on to win the election partly on the strength of his economic program, and then implement a program that turns us from the world's greatest lender nation to its greatest debtor nation in a mere eight years, we all pay the price of this nation's stupendous level of mathematical illiteracy.

Here are some possible paths for reform:

Path 1. Create mathematically solid curricula that are sufficiently enticing that they are accessible to all students, producing both a mathematically literate populace and enough well-trained students to fill our need "at the top." The arguments one hears in favor of this approach are that by expanding the pool of students to begin with, and by increasing retention rates, a high quality program that meets these literacy goals will automatically produce the quantity of high-echelon students we need. The arguments one hears against it against are that this approach is pie-in-the-sky; that a "math for all" program, whatever its virtues, will be so weak that it will fail to meet the mathematical needs of the best students.

Path 2. Try a program like that suggested in path 1, but offer either a different track or supplementary instruction for those students who are identified as "fast-track." Argument in favor: We will nurture some of those who will be our scientific and technological elite, while doing justice to the rest. This represents "elitism" in the positive sense. Argument opposed: This is first step on the slippery slope to the inherently inequitable system described in path 3. As soon as there is a fast-track system in place, ordinary kids will be shunted aside and serious attention will be reserved for the favored few.

Path 3. Identify the best students early on, and give them the kind of instruction that will hone their skills; let the chips fall where they may for the others.

In favor: This is how one makes rapid progress. Opposed: It's inherently elitist and unfair; it's unfeasible, in that one can't identify such talent early, and we would overlook some of the most talented and deepest thinkers.

(4) Raise standards to the point where a high-powered no-nonsense curriculum can be put in place and all students are expected to go through it. The best will excel and do wonders; the rest will still have a high degree of literacy. In favor: If you can get this to work, everybody wins. Opposed: This is so Pollyannaish it makes path 1 look almost plausible in comparison!

I have reasons for liking some of these ideas, and I think that some of them are disastrous for both intellectual and moral reasons; the reader may guess some of my biases. (Then again, maybe not. The one colleague who did guess when reading a draft of this article guessed wrong.) But the character of my particular biases is beside the point here. What I wish to stress here is that they *are* biases, and that both the biases and the assumptions that underlie them are very much culture-bound. More importantly, not only mine are: yours, and everyone else's are, too. The implications of this statement are explored at the end of this section. First, let me indicate that each of the paths described above is being implemented somewhere, with a good deal of support.

Some versions of path 1 are now in experimental stages around the United States, and the pro-and-con arguments summarized above rage around them. Proponents of this kind of approach can point to the small amount of research that exists about tracking. The literature appears to indicate that heterogeneous grouping helps the kids at the lower end of the performance spectrum, and does no harm to those at the top end. That is, there's no clear evidence that tracking, as it has been practiced in the U. S., has done any good (see, e.g., Oakes, 1985). Since there is documented social harm to tracking, proponents of path 1 argue that there are policy reasons for avoiding such damage if possible. Opponents note that the anti-tracking evidence is not all that strong and that the old system did succeed, as noted above, in producing our current crop of scientific and technological leaders. Moreover, because the experimental curricula are new, the data are not in regarding the potential mathematical competencies and post-instruction performance of either "average" or "high end" students who work their way through those curricula. We should go slowly, they will say, until we know what it is we're buying into.

According to David Clarke (personal communication, October 14, 1992), path 2 is the option pursued by some Australian schools: those with mathematical inclinations get more maths. This might strike some Americans as somewhat anti-democratic, but that's my point: with a different set of background assumptions, different options become possible.

Path 3, as noted above, would occasion fierce opposition in this country: it would be declared elitist, unfair, and unworkable. It may be elitist and unfair, but the fact is that it can be made to work, if one wants to make it work: In little more than a decade, for example, the "key schools" in mainland China created a cadre of world-class gymnasts where none had existed before, and the Chinese became a major force to be reckoned with in international gymnastics competitions. A comparable intellectual effort in the key schools, called the "race to the twenty-first century," has been somewhat derailed by political considerations – but that doesn't matter. The point is that underlying cultural assumptions and goals can allow such an approach to be taken – and with demonstrable success in some arenas.

Regarding path 4, readers might wish to take a look at the Japanese national college entrance exams (Wu, 1993). These examinations, required of *liberal arts students* for entrance to Japanese colleges and universities, demand a standard of mathematical literacy that many of our mathematics majors would be hard pressed to meet. Is it possible to have such standards? Yes. Is it possible for the United States to have such standards, or to adopt such a policy? It depends, you see. . . .

The phrase "it depends" is the one I wish to pursue here. In deciding whether or not to pursue any of the options listed above, we are in fact making value choices. We should understand that fact, and act with care and deliberation. Here are two illustrative examples.

Example 1. Some years ago Ralph Phillips, then chair of the mathematics department at Stanford, discussed his days in graduate school. The program he went through used a version of the Moore Method. It did him a world of good: he learned early on what it was to *do* mathematics, and he thrived. Nor was Phillips alone, of course. Moore is credited with the development of an unusually large number of world class mathematicians.

Is the Moore Method as Phillips experienced it a "good thing"? It depends on how you look at it. Phillips noted that in a class of thirty or so graduate students, he was one of the two or three who did well. The others withered under the pressure, most of them leaving mathematics as a result of their trial by fire. Presumably the world of mathematics was improved for the training Phillips received. He and others like him may have become better mathematicians, and produced better mathematics, as a result of their experiences. And, some might argue, the other students were lucky to find out early that they weren't destined to be top-flight mathematicians. However, the down side of the method by which Phillips was trained is that mathematical community lost many people who might, in other circumstances, have become productive members of it. Note that this represents not only a loss in personal terms, but a loss to the community as a whole: we should remember that (a) some very talented researchers who might have developed wonderfully in a different kind of environment left mathematics, and (b) there are lots of ways people contribute to the community beyond producing research.

Example 2, which makes the point more sharply, comes from Henry Pollak (personal communication, April 9, 1993). He wrote as follows.

> Let me make a few comments about tracking. I remember arguing about path (3) with the Ministry in [Country Z], which followed that pattern. There was a critical exam which determined whether you got a chance to go on (like the old 11+). If you were sick that day, or had to miss it for some reason, or anything, tough. When I argued how unfair this was, the answer came back that the country got enough good people by the present system, and there was no point in overtraining someone you didn't have a job for. You fill your quotas with good people, and that's all that matters.
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> We believe that everyone should go as far as their talents allow. Great. But what happens when the arrangement that allows someone else to go as far as *they* can, interferes with *your* going as far as *you* can? That's the toughie.

I could give more examples, but I hope these highlight the issue. Typically arguments about the merits of plans like the four plans mentioned above are presented in terms of absolutes. Such arguments are ill-founded. What we can do, or what we are willing to do, depends on the assumptions we make – assumptions about what is possible, about the gains likely to result, and about the costs of attaining those gains. Those assumptions represent value choices, and they are often based in cultural patterns and traditions that are taken for granted with inadequate reflection. We need to be much more thoughtful in considering issues such as tracking. In our dialogues we need to be reflective, overt, and honest about the assumptions we make.

PROBLEM-BASED CURRICULA

This section of the paper casts into relief some of the issues discussed in the opening section, "the question of content." There I discussed general issues of curriculum organization. Here I focus on one particular approach, the rationale behind it, and issues concerning what we do or don't know regarding it. The broad issue concerns using of a problem-based approach to curriculum design rather than an approach based on classical content lines. The more narrowly focused issue is hardly narrow, for it concerns *the* central issue in all learning: transfer.

In "the" problem-based approach (I use quotation marks because it is hardly monolithic), problems are the major vehicles for introducing important issues and their solutions are the major carriers of curricular weight[2]. In one sense, there is a long tradition of using problems in some such capacity. Stanic and Kilpatrick (1988) document the existence of problem manuscripts dating to 1650 B. C.; Pólya's work is legendary; and the Moore Method is widely known. But there is a difference between the approach represented in those attempts which, while problem-based, is still organized by content, and some more recent attempts in which the content may not be easily classifiable or recognizable in classical terms. That is, Pólya and Szegö's (1972) *Problems and Theorems in Analysis I* is clearly about analysis and chapters of Pólya's (1981) *Mathematical Discovery* are devoted (for example) to geometric constructions and recursion, which are clearly recognizable as (alternate organizations of) content; the generic version of the Moore Method is that students are induced to learn a more or less standard content domain by being given a collection of problems and proofs in that domain. In either case, those familiar with

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[2] Note that the problems may be either pure or applied; neither focus is implied by the phrase "problem-based."

the classical content would have little difficulty recognizing the content in those approaches.

That is not always the case with some of the more recently developed problem-based curricula. In the curriculum designed by the Interactive Mathematics Project, for example, students are introduced to linear functions in a unit called *The Overland Trail*, where a task is to plan the purchases a group of people will need for an extended trip. Aspects of trigonometric functions are discussed in a unit based on Poe's story *The Pit and the Pendulum* – the question being whether, in the story described, a person would actually have time to escape. In the standard curriculum such contexts might be used as "cover stories" to motivate a unit, and then one would get down to the "real math," as traditionally organized. But here, the solutions to the problems, in context, *are* the large part of the mathematics studied. That is, the mathematics often appears in a particular context, and aspects of it are worked out in that context; the more extended, formal presentation and decontextualization of the mathematics is not undertaken. In *Mathematics: Insight and Meaning,* Jan de Lange (1987) makes the difference clear when discussing the "realistic mathematics" materials developed by the I. O. W. O. Institute in the Netherlands, now known as the Freudenthal Institute. The extremes are far, far apart. At one end of the spectrum de Lange places materials of the Bourbaki school, which are purely formal. At the other end are the realistic mathematics materials. An illustration of this end of the spectrum is the chapter of the realistic mathematics materials dealing with elementary matrix operations. It begins with the following example entitled "Jeans."

> *Jeans come in many brands and sizes. A shop has 23 pairs of Wrangler jeans in stock.*
>
> *Sizes: 28″ (28 inch waist): 3*
> *30″ 11*
> *32″ 6*
> *34″ 3*
>
> *Other brands and sizes:*
>
> *Levis : 5, 5, 3, and 4 resp.*
> *Club de France : 1, 7, 0, and 0*

Bobos : 3, 0, 0, and 3
Ball : 3, 0, 0, and 3

> *All this information can be written down in a well-ordered matrix. . .*
> (de Lange, 1987, p 109).

Students are asked questions about the total number of particular items such as Levis or pairs of pants with a 32" waist (questions which require computing the sums of rows or columns of the matrix), about the total stock if one is given matrices that document the number of purchases and sales for the next week (computed by matrix addition and subtraction), and about the total value of the inventory if the costs of each item are given (computed by matrix multiplication). The mathematics is introduced and used in an applied context, and it is not presented in its abstract form. Readers who expected to see standard definitions of matrix operations did not find them in early editions of the material, although they were added later at teachers' request. One still does not find problems like the ones found in standard texts, such as:

Solve for x_1 and x_2:

$$\begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix} \bullet \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

The development of curricula of this type raises a host of questions. I shall focus on three, albeit briefly: (i) the role of the problems and the uses to which they are put; (ii) the issue of content versus face validity; (iii) the deep question of transfer as it relates to learning and doing mathematics.

(i) Role and uses. The issue of problem selection is of critical importance. I spent a full year in the mathematics library at Berkeley selecting the problems to be used in my first problem solving course, and in the welter of problem books and other sources, I found that about one problem in every 500 struck me as being a good problem for the course. Over the years I have come to understand my selection criteria, which I refer to as a "problem aesthetic." The following description of that aesthetic is modified from Schoenfeld (1991).

I want the problems my students work on – whether in problem solving courses or in regular instruction – to serve as introductions to mathematical thinking. The problems I use in my courses, and that I am always on the lookout for, tend to have the following four properties.

A. In general, good problems are (relatively) accessible. I like problems that are easily understood and that do not require a lot of vocabulary or machinery in order to make progress on them. Note that these criteria do not mean the problems are easy or conceptually trivial: undergraduates can start work on the Four Color Theorem and Fermat's Last Theorem without knowing too much background mathematics!

B. For many reasons, I tend to prefer problems that can be solved, or at least approached, in a number of ways. It's good for students to see multiple solutions, since they tend to think, on the basis of prior experience, that there is only one way to solve any given problem (which is usually the method the teacher has just demonstrated in class). I need for them to understand that the "bottom line" is not just getting an answer, but seeing connections. Moreover, on the process level, the possibility of multiple approaches lays open issues of executive decisions – what directions or approaches should we pursue when solving problems, and why? (Executive decisions often make or break a problem solving attempt. See Schoenfeld (1985) for details.)

C. The problems and their solutions should serve as introductions to important mathematical ideas. This can happen in (at least) two ways. Obviously, the topics and mathematical techniques involved in the problem solutions can be of agreed importance. Equally important, the solutions to the problems can illustrate important problem solving strategies, and serve as a "training ground" for the students' development of strategic skills.

D. Problems used in my course should, if possible, serve as seeds for honest-to-goodness mathematical explorations. Open-ended problems such as the following, for example, provide one way to engage students in *doing* mathematics:

> We all know the Pythagorean theorem, which says that if A and B are the lengths of the legs of a right triangle in the plane and C is the length of its hypotenuse, then $A^2 + B^2 = C^2$. Let's start with that. Can you prove the theorem? In how many different ways? Can you extend it, or generalize it?

> You know typical whole-number solutions, e.g. the (3, 4, 5) right triangle. Are there others? Can you find them all? What else?

I try to choose problems that are extensible and generalizable. Good problems lead to more problems – and if the domain is rich enough, students can start with the seed problem and proceed to make the domain their own. (One example of a simple start-up problem, discussed in Schoenfeld (1990), asks students to fill in a 3 x 3 magic square. That problem is trivial, but its extensions and generalizations have occupied many mathematicians for years.)

So, I have my problems and my intended purposes. Others have theirs, and the purposes may vary – e.g., others, given their goals, may have a more content-oriented focus than I tend to have; or some problems may be intended for recreation, and others as introductions to neat ideas, without there being an intention to follow up in detail. Once you have the problems, what do you do with them? Are there absolute rights and wrongs? I am tempted to say "no" – at least, as long as one is reasonably sensible. That is, given any reasonable behavior (e.g., "telling the answer"), I can imagine circumstances where that is precisely the right thing to do (from my point of view), and other circumstances where it would be precisely the wrong thing to do. The point I want to stress here – and it may discomfit those who believe that there are absolute rules for good teaching, and that those rules should be followed – is that few such rules exist beyond the very obvious ones. Good pedagogical decision-making is often context-bound, and a large part of what is "right" or what "works" is a matter of personal style and preference.

Let me be careful here, to minimize the chance of misinterpretation. Of course there are lots of bad things one can do in the classroom; indeed, a significant part of my research has been spent uncovering the unfortunate consequences of well-intended but less-than-optimal instruction. And there *are* absolutes, obvious rules like "respect your students and act accordingly." What I am saying is that, within the realm of good instruction, there is significant variance – and that some things that some teachers can really make work for their students may make other equally successful teachers feel quite uncomfortable. For example, should one bring lessons to closure, and if so, when? Recently I watched a videotape of a lesson taught by someone who was obviously a master teacher. The class ran wonderfully without much apparent direction from the teacher (a testament to the teacher's skill), and the focus was on real mathematical ideas. In that lesson there was a wideranging discussion and no apparent closure: the mathematical issues were left hanging, and I couldn't tell from the videotape when or even whether they *would* be resolved. Had I been teaching that lesson, I certainly would have said something to bring things to closure. Indeed, I had to restrain myself from talking at the tape! But I noticed that there were numerous places in the lesson where I would have been tempted to intervene, and where subsequent evidence in the tape indicated that the teacher's non-interventiveness had resulted in the students' producing interesting and worthwhile mathematics. And, I imagine, there are places in other lessons where the teacher *does* step in – perhaps before I would have.

Should one give problems whose solution lies beyond the students' current reach? Yes, sometimes – if the problems pique the students' curiosity and the students can make some legitimate progress on them. Should one give the answer to problems, and if so, when? I would tend to say "most of the time" to the first, but the question of when is very much an open one. Above I mentioned the problem of finding integer solutions to the diophantine equation $A^2 + B^2 = C^2$ as an example of a problem I like. There is a well-known solution to the problem, which would take me about five minutes to present in class. In a lecture class on number theory, that's what's typically done. I could do the same in my problem solving classes. However, doing so would have precluded the discoveries made by my students in one class: that there are infinitely many relatively prime Pythagorean triples of the form
(x, y, y+1), infinitely many of the form (x, y, y+2), and none of the form (x, y, y+3). In making these discoveries (which are minor, but were new to me and many of my colleagues), the students were *doing* mathematics, and that was a big payoff for my delaying the revelation of the fully known answer. Now, I *did* feel constrained to tell them about the answer; I felt it would be inappropriate not to, since the result is known, and its existence in no way diminishes their achievement. But there have been times when I would only say "there is a known solution, but I don't want to discuss it; you can look for it if you want, or better, keep working on it," and there have been times when I couldn't tell the students because I didn't know myself: I might have brought a problem to the class because I thought it was interesting and thought we could all work on it, or one might have emerged in the classroom discussion. What are the absolutes? You must listen to your students, use discretion and common sense, make your judgments based on your understanding of your goals for instruction whenever possible, and always be honest.

(ii) Content. This is where push comes to shove when one views what is "in" a problem-based curriculum, and what students are likely to take from it. The issue of content may have been the most contentious issue we faced (tracking included) when producing the 1992 California *Mathematics Framework*. Some of the dilemmas are these.

On the one hand, the familiar packaging of content – "scope and sequence" lists, or examples like the one given earlier, "Limits and rate of change, 4 lectures . . ." are accessible and have a good deal of tradition behind them. Those who grew up in that system tend to think of curricular content in those terms. It is natural and appropriate for them to ask, when viewing a proposed curriculum, "Where will students learn to factor quadratics? What facility will they develop with manipulating algebraic objects, or solving various kinds of equations? Where will they learn the definitions of trig functions, and where will they develop flexibility operating on them?" And so on. And if those who propose a new curriculum do not have at least some ready answers to those questions, there are grounds for concern. Indeed, those were the grounds for concern in the California case. We were not proposing curricula, but frameworks for acceptable curricula; if we didn't specify the mathematical content to be found, what guarantees would there be that it was there?

Here were the major concerns on the opposite side of the argument. First, the content description so dear to the hearts of those who propose it is in itself a red flag to others. Some reformers argue that the list of topics in Algebra I (for example) is *precisely* what's wrong with the content of Algebra I. The course itself consists of a sequence of decontextualized, often sterile, techniques and topics, that are completely divorced from meaning or applications. They are the tools, but not the curriculum, in the same way that listing "lathe, band saw, orbital sander, . . ." is a list of tools for a woodworking class, and not of the kinds of things students will be expected to produce. The historical curriculum focus has been on such lists of topics; it has ignored issues of problem solving strategies, metacognition, belief, mathematical culture. Why use the structure of a failed system to constrain a new one? And, the old familiar checklists put people in the mind set of the old, familiar curriculum. This has a stifling effect: from the old mind set the easiest way to make sure X is in the new curriculum is to localize instruction about X in a unit that teaches X, and this is what reformers are trying to avoid.

Underlying this conflict is an honest-to-goodness dilemma. Over the past two decades we have made real progress in characterizing, at the process level, some of the desired outcomes of a successful mathematics education. However, partly because the issue is as complex as it is, we have nothing vaguely resembling a functional description, in process-plus-content terms, of what we'd like high school graduates, college-intending students, or intending mathematics majors to know.[3] We have few if any appropriate assessment measures or techniques, though we are working in that direction. In the United States we have no real existence proofs – that is, there do not exist substantial numbers of students who have gone through the reform curricula and emerged demonstrably competent to do further work either in collegiate mathematics or in the workplace. Current reform programs haven't been in place long enough to produce them; moreover, whether it is justifiable or not, prior reform efforts such as the "new math" or "discovery math" have been branded failures, and the current reform effort is tarred with that brush. The old system, while it may not have worked very well in a deep sense, did work in the sense that it provided yardsticks that functioned within the system. (Never mind that only in the very crudest and quite likely inaccurate sense did it provide a description of what we'd like high school graduates, college-intending students, or intending mathematics majors to know. It supported a content measurement scheme that "worked," in the sense that people who did well at level N tended to do well at level N + 1.) Why abandon the old content specifications, some would say, until you can replace them with something that is demonstrably better? The more conservative among us want to save the rest of us from going off the deep end.

The fact is that, in the absence of either large-scale empirical proof of success or the existence of compelling and documentable standards, there *is* reason to be cautious. The traditionalists are nervous for good reason. It should be noted, however, that the resistance to change is not based on the purported success of current curricula (one is hard-pressed to find people who say that we are doing things well!), but on the fear that the replacement will be even worse. Here it is worth returning to the notion of a zero-based curriculum planning process.

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[3] The *Standards* are a vector in the right direction, but we should not make the mistake of thinking that they delineate in anything resembling operational terms (a) what students need to know, or (b) how to know if they know it.

Suppose we declared that any proposed curriculum must, in order to be implemented, make a plausible case that it would do well. The reform curricula would fail because they cannot yet produce real proof, or real standards. But current mainstream curricula would fail even more strongly because there exists a massive body of evidence indicating that they do not work. Conclusion: we can not and must not inhibit the extensive field testing of well designed reform curricula, but we must at the same time be vigilant. Good ideas – functionally, ideas that receive the endorsement of a significant proportion of the knowledgeable community – should be tried, but the trials should proceed, as suggested in the section on content, with the highest standards of honesty.

(iii) Transfer. Here I want to remind us all how little we know about some fundamental issues. Since some of my current research takes place in the context of Algebra I instruction, I will use that as an example.

Much of the power of algebra (and of mathematics in general) is that of abstraction: once one has the appropriate symbolic representation of a situation, one can operate on the symbols and derive relevant information from them. To pick two familiar examples, the age and mixture problems found in elementary algebra curricula typically give rise to systems of two equations in two unknowns. Once the equations have been obtained, one can solve them without reference to where the equations came from (until one needs to map the answer back to the originating contexts). That is, the symbolic manipulations can be done context-free, and it is that fact that yields much of their power. The result, in standard pedagogical practice, has been that we teach the context-free symbol manipulations in the hope that students will be able to apply them – first in the artificial contexts of word problems and then, one hopes, in more realistic contexts. The results of this approach have been, shall we say, less than impressive. A small percentage of the students (myself and my mathematical colleagues included) "get it"; the rest don't, and fall by the wayside at the rate of 50% per annum from grade nine onwards.

The flip side of the argument, as expressed in the Dutch work (de Lange, 1987) and others', is to say that the mathematics should be learned in context to be meaningful. There is evidence that students can learn fairly high-powered mathematics in particular contexts – e.g., secondary students can use matrices to do inventory problems and use recursion relationships such as the one that generates Fibonacci sequences to deal with the growth of hypothetical rat populations. This is impressive, but it is only a first step. Having learned the mathematics in one context, will the students be able to deal with it out of context? That is, will they be able to do the abstract matrix manipulations or recursions, when they do not arise in the context of inventory or population growth problems? Having learned the mathematics in one context, will they be able to recognize when other situations are essentially isomorphic, and that the same mathematical analysis applies? This, of course, is the "transfer problem." We understand little about it, but we do know that it's hard enough that one shouldn't be sanguine about the answers to the questions just raised. My work on heuristic strategies suggests that students may have a good shot at learning both a mathematical idea and its application in various contexts if they first learn it in one context, then another, perhaps another, then have their attention drawn to the common features, then look for the common features in a fourth context, and so forth. We are trying this idea in a unit on linear functions, which is seeing the light of day in local classrooms as I write these words. We plan to look at what happens very closely, both in the hope of making progress on understanding transfer and of designing a piece of curriculum that really works. I do not know of other groups that are taking this close a look at the issue of transfer, and that concerns me: It's very hard, and lies at the core of successful mathematics learning. As an intellectual community we need to focus on theoretical issues alongside pragmatic issues, so that progress on each can inform work on the other.

DO WE NEED PROOF IN SCHOOL MATHEMATICS?

Absolutely. Need I say more? Absolutely. Proof is one of the most misunderstood notions of the mathematics curriculum, and we really need to sort it out. What is it, what roles does it play in mathematics and mathematical thinking, and how and when can students learn to deal with it?

There are, I think, three roles of proof that need to be explored and understood: the unique character of certainty provided by air-tight mathematical arguments, which differs from that in any other discipline and is part of what makes mathematics what it is; the fact that proof need not be conceived as an arcane formal ritual, but can be seen as the mere codification of clear thinking and a way of communicating ideas with others; and the fact that for mathematicians, proving is a way of thinking, exploring, of *coming to understand* – and that students can and should experience mathematical proving in the same ways.

The first aspect of proof is a part of our deep mathematical culture and heritage, and I need say little about it. One of the glorious things about proof is that it yields certainty: When you have a proof of something you know it *has to be* true, and why. That feeling of certainty is really powerful, for patterns and trends can be deceptive. All mathematicians have their favorite examples of patterns that look like they ought to hold but fail, or of conjectures that are true for the first N tries but then fail. The most extreme case I know: When N is an integer, is $(1 + 1141N^2)$ a perfect square? On the basis of reasonable empirical evidence, say the first couple of trillion cases, it looks like you could bet the mortgage that it won't be. In fact, $(1 + 1141N^2)$ isn't a square for N = 1, 2, 3, . . . , 30,693,385,322,765,657,197,397,207 – but it *is* for N = 30,693,385,322,765,657,197,397,208. (Oops!) Here's a simple example in contrast. Take any odd number. Square it, and then subtract 1. I bet that the number that's left will be divisible by 8, without leaving a remainder. And you *can* bet the mortgage on that one. Why? Say the number you choose to begin with is M. Since M is odd, it can be written as (2k + 1), where k is an integer. Then

$$M^2 - 1 = (2k + 1)^2 - 1 = 4k^2 + 4k = 4(k)(k + 1).$$

This tells us that $(M^2 - 1)$ has a factor of 4. To find an additional factor of 2, we focus on k. If k is even, then it has factor of 2; and if k is odd, then (k + 1) is even, so that term provides the additional factor of 2. Either way (and those are the only two possibilities, of course), $(M^2 - 1)$ has no choice but to be divisible by 8. So, pick an odd number, any odd number – say N = 30,693,385,322,765,657,197,397,207, because it happens to be close at hand. I don't have to carry out the calculation; I *know* that if you square it and subtract 1, the result will be divisible by 8. And I know why. I think that's beautiful.

Why do our students have so little appreciation for proof, and so little apparent aptitude for it? It is, I think, largely our fault; the way students encounter proof in school, it takes oddballs like us to appreciate it. In most instructional contexts proof has no personal meaning or explanatory power for students. Consider a course in Euclidean geometry, for example. Typically, in their perception, we ask students to prove only that which is already intuitively obvious to them – e.g., that base angles of an isosceles triangle are equal. They are well aware that millions of students have proved the same things before them – hence proof becomes a procedure for confirming what is already known to be true. It may be "good for you" in the same way calisthenics are, but it really doesn't connect to issues of personal intellectual importance. (See Schoenfeld (1988) for an extended discussion of this issue.) In addition, although it wasn't intended this way, there has been a tremendous emphasis on *form* that tends to override issues of substance. In a conversation in one of my videotapes, for example, two students produce an absolutely beautiful mathematical argument that provides the proof that they were trying to find. They know they have understood the problem completely, but at that point one of them says: "We're not being mathematical about this." They then spend ten minutes laboriously converting their lucid and correct mathematical argument into the standard, clumsy two-column format for Euclidean proof. My classroom observations suggest that this kind of behavior is the result of an overriding classroom emphasis on form over content: Students believe that proof-writing is a ritual to be engaged in, rather than a productive endeavor.

Finally, I note that proof has at least two important senses in mathematics: the finished product, which is what I have been discussing up to this point, and the process by which the finished product is obtained. If mathematicians want to know if something is true and why, they may look for examples to develop some intuition, and then try to prove the result. The attempt to produce a proof is an attempt to explore structure: "If this happens, then this works this way, and so these pieces should fit together that way. . . ." In the process of trying to produce a proof, mathematicians find out what make the objects under study "tick." Trying to prove, or disprove, is a form of exploration.[4] One does that for oneself, for that is how one comes to understand a piece of mathematics; one writes it down, for that is how one communicates one's understandings. Proof is not a thing separable from mathematics, as it appears to be in our curricula; it is an essential component of doing, communicating, and recording mathematics. And I believe it can be embedded in our curricula, at all levels.

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[4] Here it may be worth going back to the etymological roots of "proof." The roots of "prove" are in the Latin *probare,* defined in the Oxford English Dictionary as "to test (a thing) as to its goodness." When mathematicians try to produce a proof of something they suspect is true, that is precisely what they are doing: trying to see if they can show that it works (and how). As long as I'm digressing, I should point out that this sense of "prove" is the one originally meant in the phrase "the exception proves the rule": the exception *tests* the rule and finds it wanting, rather than verifying it.

Now, some practical comments. As noted above, I think of proving as a codification of good thinking – a clear explanation of why something is the case – rather than as something special. Recently I watched a videotape of a class of third-grade students discussing the properties of odd and even numbers. The classroom discourse was astounding: ideas were put forth as conjectures, discussed, and defended or rejected on pretty solid mathematical grounds. If some students' arguments had been written up in prose form, they would have been proofs that mathematicians would accept. And I think the students in that class would accept the proofs and think they are important, because they represent meaningful arguments developed by the students in an attempt to understand something (as opposed to formal trappings divorced from meaning). I think that if students grew up in a mathematical culture where discourse, thinking things through, and *convincing* were important parts of their engagement with mathematics, then proofs would be seen as a natural part of their mathematics (Why is this true? It's because . . .) rather than as an artificial imposition. In that context, I can't improve on the sequence suggested by Mason, Burton, and Stacey (1985, p. 95) in moving from intuition to proof, as a form of understanding and communication:

> Convince yourself.
> Convince a friend.
> Convince an enemy.

Of course, we don't start from scratch with the current population, so establishing a climate where proof is perceived as useful is more difficult than it might otherwise be. Just for the record, here are two devices I use in my problem solving course. (I make no claims for originality; these are widely known.)

Device 1. Over the period of a few days I have students work a sequence of problems including the following:

1. If S is any set, we define N(S) to be the number of subsets of S, including the null set, ø. For example, if S is the set {A, B, C}, the subsets of S are: ø, {A}, {B}, {C}, {A, B}, {A, C}, {B, C}, and {A, B, C}, so N(S) = 8. Let S be a set of 99 elements. What is N(S)?

2. What is the sum of the coefficients of $(x + 1)^{37}$?

3. Suppose you pick 62 points on the boundary of a circle. You then draw all of the line segments that connect pairs of those points. If the points have been chosen so that no three of the segments intersect at the same point (that is, the circle is divided into the maximum possible number of regions), into how many regions is the circle divided?

These are embedded in a collection of problems given toward the beginning of the term. One of that collection's "morals" is that it can be useful to do empirical explorations: if you do a few examples, you might see a pattern. By the time they work these problems, they have learned to do so. On Problem 1, they see that the pattern is that a set of N elements has 2^N subsets. I ask if they're convinced. "Yes." "Are you sure?" "Yes." "Do you want to prove it?" "No." "Humor me." With some grumbling they do. A similar discussion takes place for Problem 2. And, likewise, a similar discussion begins for Problem 3: the number of regions one gets for N = 1, 2, 3, 4 and 5 is 1, 2, 4, 8, and 16 respectively. At this point most of the students are convinced the pattern will be 2^{N-1}. I ask if they're convinced. "Yes." "Are you sure?" "Yes." "Do you want to prove it?" "No." "Well, try another case or two and see if you can prove it." It turns out that for N = 6, the number of regions is 31. At first the students refuse to believe it; they re-draw and re-count. When they are convinced the pattern doesn't hold up, we have the grist for a conversation.

Device 2. More important, I think it is essential that students engage in mathematics where proof is meaningful and useful for them. One of the loveliest instances of this comes in the collection of geometric construction problems given in Chapter 1 of Pólya's (1981) *Mathematical Discovery.* If you try to solve construction problems like this one:

> You are given two line segments of length **a** and **r**, respectively, and an angle of measure α. Construct a triangle that has the following properties:
>
> i. One side of the triangle has length **a.**
> ii. The radius of the inscribed circle of the triangle is **r.**
> iii. The measure of the angle opposite the side of length **a** is α.

you will quickly discover that you are making conjectures and proving things, along the way to finding the properties you need to do the construction. Pólya's problems start off easy, and get tough. Students can really get involved in them, and get involved in the mathematics required to do them. One of my fondest classroom memories is of a student raising his hand after we'd worked a few such problems and asking, "Are you trying to tell us that proof is good for something?"

In Schoenfeld (in press) I describe some of my attempts to create a classroom atmosphere in which reasoned mathematical discourse is the norm, and proof a natural means of exploration and communication. This might seem pie-in-the-sky, but the videotape of the third-grade classroom I watched the other day, and many others, convince me that it is possible to have mathematics classes be communities in which mathematical sense-making takes place. And when that happens, proof will be a necessary component of the sense-making and discourse processes. I do not claim that this will be easy. But it is a critically important goal, and the effort expended in understanding what it takes to make it happen will be well worth the trouble.

CODA

These are exciting times. Over the past two decades we have made spectacular strides in understanding the nature of mathematical thinking, teaching and learning; we have the beginnings of a knowledge base that should allow for much more meaningful, enjoyable, powerful, and empowering mathematics instruction. We are also fortunate that there is a social impetus for curriculum change (though this is hardly an unalloyed blessing, given the ludicrous high-stakes accountability pressures that accompany it!), so that we may be able to put some of these ideas into practice. We should be proud of what we now know, and glad of the opportunity to move ahead – but we should be humble because of the huge amount we do not know, and we should proceed with caution. Some of the ideas we work with can be taken as givens. For example, the "constructivist perspective" is better grounded in empirical and experimental evidence than the theory of evolution; we should just assume it and get on with our business (while working damned hard, of course, to flesh it out and understand it more fully). But many other things are a matter of informed judgment – untested with large numbers of students, weakly grounded in detailed defensible research, and anything but a sure bet. Let us proceed; let us keep our eyes open for our successes and our failures, and be ready to learn from both. As Pólya puts it so eloquently in *Mathematics and Plausible Reasoning* (1954, page 8): "Intellectual courage, intellectual honesty, and wise restraint are the moral qualities of the scientist."

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